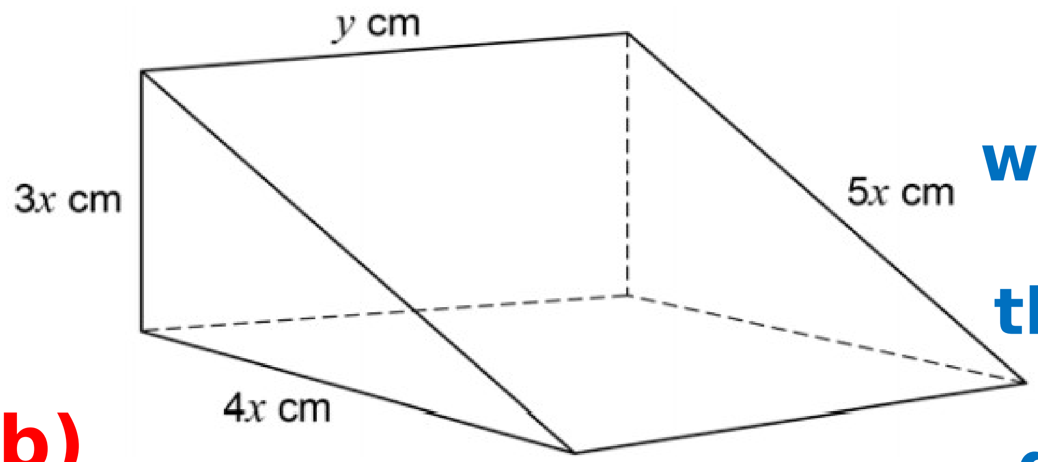
Tuesday, May 27, 2025

4.5 Turning Points - Applications

The diagram shows a block of wood in the shape of a prism with a triangular cross-section. The end faces are right-angled triangles with sides of lengths $3x$ cm, $4x$ cm and $5x$ cm, as shown in the diagram.

You can often do part (b) without part (a) with these types of questions.



Do part (b)

The total surface area of the five faces is 144 cm^2.

(a) Show that the volume of the block, V cm^3, is given by $V = 72x - 6x^3$

(b) Show that V has a stationary value when $x = 2$ and determine whether it is a maximum or a minimum.

Fully justify your answer.

G Differentiation

G3	Apply differentiation to find gradients, tangents and normals, maxima and minima and stationary points, points of inflection. Identify where functions are increasing or decreasing.

https://sites.google.com/view/tlmaths/home/a-level-maths/as-only/g-differentiation/g3-gradients#h.84gr8x83bccr

G3-24 through to G3-27

G Differentiation

Students should:

- understand and be able to use the fact that at a stationary point, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$
- describe a stationary point as a (local) maximum or minimum
- know that:

  At a maximum $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$

  At a minimum $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$

  Note:

  the case $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ will not be tested at AS

- use $m_1 \times m_2 = -1$ for gradients of tangent and normal
- be able to answer questions set in the form of a practical problem where a function of a single variable has to be optimised
- be able to show that a function is increasing or decreasing, by showing $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$ or $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$ respectively.

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Optimisation

Since we can use differentiation to find the maximum or minimum of a function, it can be used in real-life problems to maximise or minimise a certain quantity subject to certain factors.

For example:

- Maximising the profit given supply/demand constraints
- Minimising the surface area of an object subject to a specific volume

4.5 Turning Points - Applications

Example 1a

Figure 4 shows a solid brick in the shape of a cuboid measuring $2x$ cm by x cm by y cm.

The total surface area of the brick is 600 cm^2.

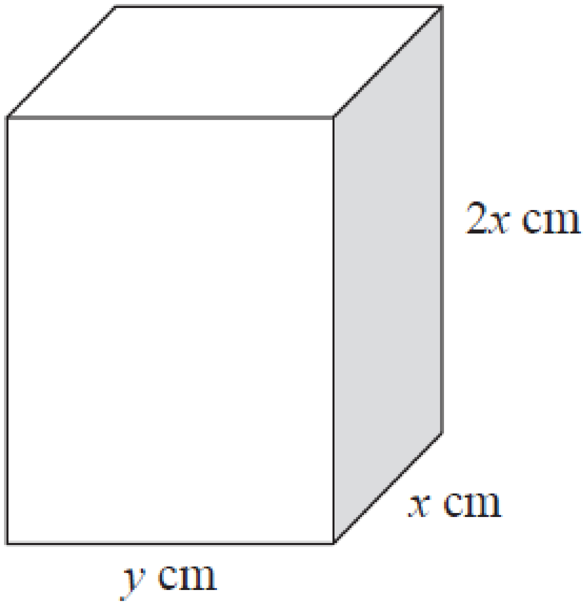


Figure 4

(a) Show that the volume, V cm^3, of the brick is given by

$$V = 200x - \frac{4x^3}{3}.$$

Surface Area:

Volume:

4.5 Turning Points - Applications

Example 1b

Given that x can vary,

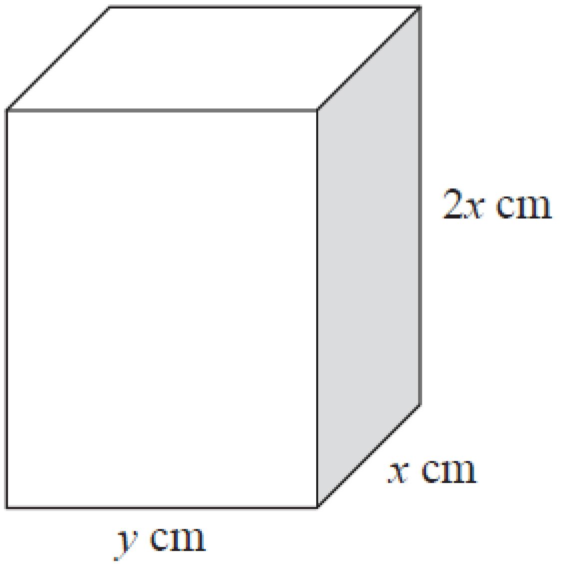
(b) use calculus to find the maximum value of V, giving your answer to the nearest cm^3.

(5)

(c) Justify that the value of V you have found is a maximum.

(2)

cm



4.5 Turning Points - Applications

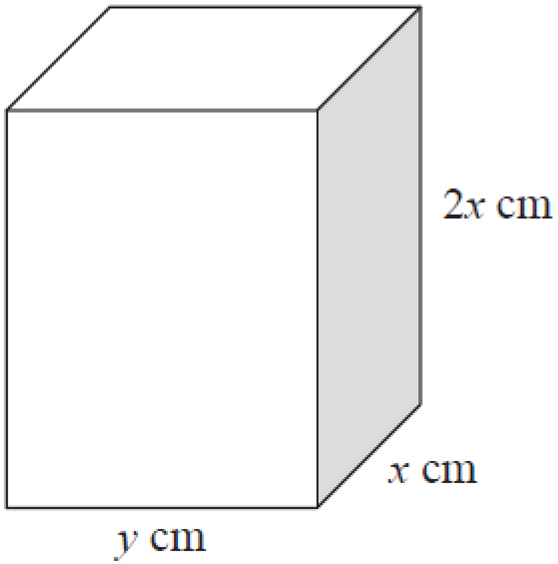
Example 1c

Given that x can vary,

(b) use calculus to find the maximum value of V, giving your answer to the nearest cm^3.
(5)

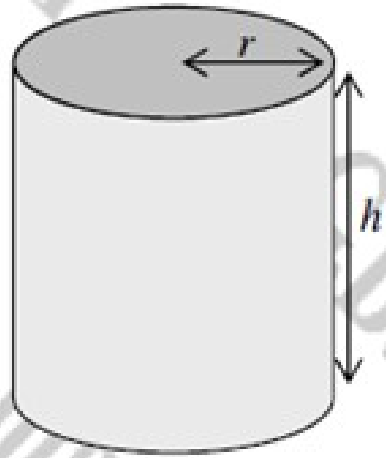
(c) Justify that the value of V you have found is a maximum.
(2)

the value of V is a maximum



Example 2a

You need to know the formulae for volume and surface area of cylinders and other solids!



Notice that the area is given in terms of r only, h has been eliminated

The figure above shows a **closed** cylindrical can, of radius r cm and height h cm.

a) If the volume of the can is 330 cm^3, show that surface area of the can, $A\text{ cm}^2$, is given by

$$A = 2\pi r^2 + \frac{660}{r}.$$

Volume of a cylinder:

Surface area of a closed cylinder:

Example 2b

b) Find the value of r for which A is stationary.

c) Justify that the value of r found in part (b) gives the minimum value for A.

d) Hence calculate the minimum value of A.

Stationary points when

$$r = \sqrt[3]{\frac{165}{\pi}} = 3.745$$

Example 2c

b) Find the value of r for which A is stationary.

c) Justify that the value of r found in part (b) gives the minimum value for A.

d) Hence calculate the minimum value of A.

$$\frac{dA}{dr}=4\pi r-660r^{-2}$$

$$r=\sqrt[3]{\frac{165}{\pi}}=3.745$$

the value of gives the minimum for

Example 2d

b) Find the value of r for which A is stationary.

c) Justify that the value of r found in part (b) gives the minimum value for A.

d) Hence calculate the minimum value of A.

$$A = 2\pi r^2 + \frac{660}{r} \qquad r = \sqrt[3]{\frac{165}{\pi}} = 3.745$$

$$\therefore A_{min} = 2\pi\left(\sqrt[3]{\frac{165}{\pi}}\right)^2 + \frac{660}{\left(\sqrt[3]{\frac{165}{\pi}}\right)} = 264\, cm^2\, (3\, sf)$$

4.5 Turning Points - Applications

Method:

To *optimise* a given situation:

1. Set up **two** equations from the information in the question (we will be told the value of one equation and the other one is the variable we wish to maximise/minimise).
2. Rearrange the known equation so that the unwanted variable is the subject.
3. Substitute this equation into the equation we wish to optimise so we have it in terms of one variable.

4.5 Turning Points - Applications

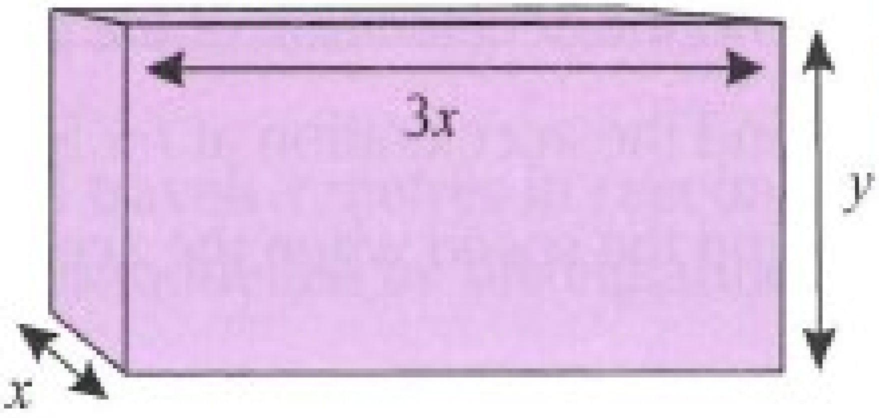
Method:

To *optimise* a given situation:

4. Differentiate this expression then find the value of the variable at the stationary point.
5. Use the second derivative to confirm that this value is a maximum or minimum.
6. If required, find the corresponding maximum/minimum volume/area/cost etc.

Example 3: not always broken down

A cuboid jewellery box with a lid has dimensions $3x$ cm by x cm by y cm, and its sides are modelled as laminas. It is made using a total of 450 cm^2 of wood.



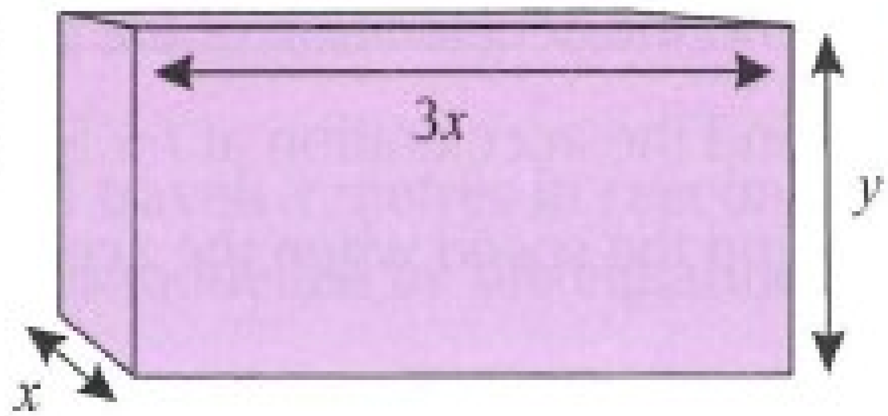
Show that the volume of the box can be expressed as $V = \dfrac{675x - 9x^3}{4}$, and use calculus to find the maximum volume.

urface area:

Volume: $V = 3x^2 y$

Example 3: not always broken down into parts!

A cuboid jewellery box with a lid has dimensions $3x$ cm by x cm by y cm, and its sides are modelled as laminas. It is made using a total of 450 cm^2 of wood.

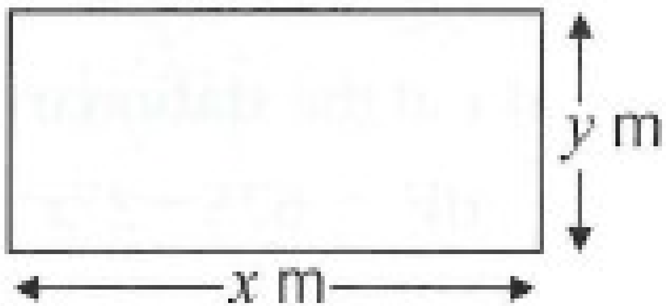


Show that the volume of the box can be expressed as $V = \frac{675x - 9x^3}{4}$, and use calculus to find the maximum volume.

Example 4: not always about surface area & volume!

A farmer wants to build a rectangular sheep pen with length x m and width y m. She has 20 m of fencing in total, and wants the area inside the pen to be as large as possible. How long should each side of the pen be, and what will the area inside the pen be?

Perimeter:

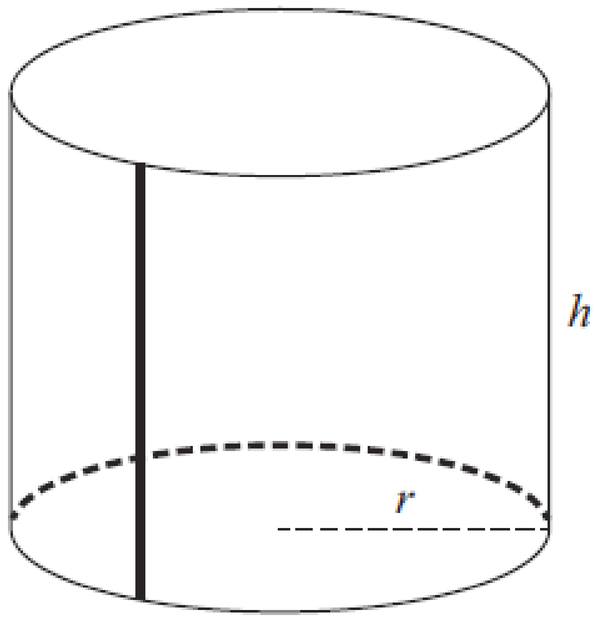


Area:

Each side should be giving a maximum area of

11 Rakti makes open-topped cylindrical planters out of thin sheets of galvanised steel.

She bends a rectangle of steel to make an open cylinder and welds the joint. She then welds this cylinder to the circumference of a circular base.



The planter must have a capacity of 8000 cm^3

Welding is time consuming, so Rakti wants the total length of weld to be a minimum.

Calculate the radius, r, and height, h, of a planter which requires the minimum total length of weld.

Fully justify your answers, giving them to an appropriate degree of accuracy.

[9 marks]

Q	Marking Instructions (I)	AO	Marks	Typical Solution (using r)
11	Obtains correct weld length in terms of h and r	AO1.1b	B1	Length of weld $= w = h + 2\pi r$ Volume $= 8000 = \pi r^2 h$
	Obtains formula for h in terms of r or vice versa using volume = 8000	AO3.1b	M1	So $h = \frac{8000}{\pi r^2}$
	Substitutes to get weld length in terms of one variable, obtaining correct formula for w	AO1.1b	A1	$w = \frac{8000}{\pi r^2} + 2\pi r$
	States that for a stationary point the first derivative is zero. (OE)	AO2.4	E1	For minimum length of weld $\frac{dw}{dr} = 0$
	Differentiates correctly (FT provided formula includes negative powers) (accept numerical value of $-\frac{16000}{\pi}$ used)	AO1.1b	B1F	$\frac{-16000}{\pi r^3} + 2\pi = 0$ leading to $\pi^2 r^3 = 8000$
	Solves to find a value of r and a value of h (do not award if the final value of r or h is negative)	AO1.1a	M1	$r = 9.32$ cm $h = 29.3$ cm
	Obtains r = 9 or 9.3 (AWRT) and h = 29, 29.3 (AWRT) or 30	AO1.1b	A1	
	Differentiates 'their' first differential and substitutes in 'their' value of r or h	AO1.1a	M1	$\frac{d^2w}{dr^2} = \frac{48000}{\pi r^4}$
	Sets out a well-constructed mathematical argument, using precise statements throughout to find the values of r (9 or 9.3) and h (29, 29.3 or 30) and justifies the minimum value. Can be awarded if E1 not obtained.	AO2.1	R1	which is positive, so this is a minimum for w
	Total		9	

A company makes drinks containers out of metal.

The containers are modelled as closed cylinders with base radius r cm and height h cm and the capacity of each container is 355 cm^3

The metal used

- for the circular base and the curved side costs 0.04 pence/cm^2
- for the circular top costs 0.09 pence/cm^2

Both metals used are of negligible thickness.

(a) Show that the total cost, C pence, of the metal for one container is given by

$$C = 0.13\pi r^2 + \frac{28.4}{r}$$

(4)

(b) Use calculus to find the value of r for which C is a minimum, giving your answer to 3 significant figures.

(4)

(c) Using $\dfrac{d^2C}{dr^2}$ prove that the cost is minimised for the value of r found in part (b).

(2)

(d) Hence find the minimum value of C, giving your answer to the nearest integer.

(2)

a) $V = \pi r^2 h$

$\pi r^2 h = 355$

$h = \dfrac{355}{\pi r^2}$

$C = 0.04(\pi r^2) + 0.04(2\pi r h) + 0.09(\pi r^2)$

$= 0.13\pi r^2 + 0.08\cancel{\pi r}\left(\dfrac{355}{\cancel{\pi} r^{\cancel{2}}}\right)$

$= 0.13\pi r^2 + \dfrac{28.4}{r}$ as required

b) $\dfrac{dC}{dr} = 0.26\pi r - \dfrac{28.4}{r^2}$

$\dfrac{dC}{dr} = 0$ at a minimum

$\therefore 0.26\pi r - \dfrac{28.4}{r^2} = 0$

$r^3 = \dfrac{1420}{13\pi}$

$r = \sqrt[3]{\dfrac{1420}{13\pi}}$ (≈ 3.26)

c) $\dfrac{d^2C}{dr^2} = 0.26\pi + \dfrac{56.8}{r^3}$

$\left.\dfrac{d^2C}{dr^2}\right|_{r=\sqrt[3]{\frac{1420}{13\pi}}} = \dfrac{39}{50}\pi \approx 2.45 > 0$ Hence minimum value

d) $C = 0.13\pi(3.26^2) + \dfrac{28.4}{3.26}$

$C = 13$ pence (nearest integer)